

# An Overview of Recent Progress at NNLO

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# Outline

The last few years have seen remarkable progress in computing higher loop corrections to Standard Model processes.

- Status of higher loop computations.
- Specific computational progress of the last few years:
  - (a) Description of improved loop integration methods.
  - (b) Two-loop calculations with multiple kinematic variables.
- Applications:
  - (a) Higgs signal and backgrounds at the LHC.
  - (b) Example of theoretical study:  $N = 4$  super-Yang-Mills.
  - (c) Promise for future.

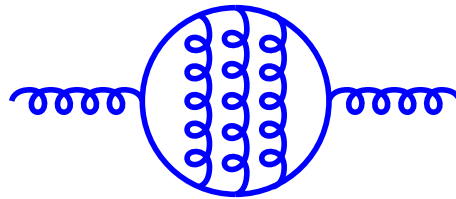
## Motivation for NNLO

- Reduce QCD renormalization scale dependence.
- Reduce mismatch between parton-level and experimental hadron level jet algorithms.
- Cleaner separation of perturbative and non-perturbative power contributions.
- Improved description of final state transverse momentum due to double radiation from initial state.
- NNLO global fits to pdf's should help reduce theoretical uncertainty.
- Electroweak NNLO also needed.

# Status of Higher Loop Computations

Some examples of well known impressive higher loop computations:

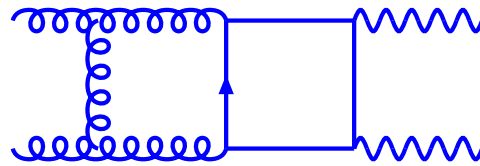
- $g - 2$ , 4 loops, Kinoshita and many friends.
- $R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ ,  $O(\alpha_s^3)$  Gorishny, Kataev and Larin, etc
- 4-Loop QCD  $\beta$  function van Ritbergen, Vermaseren and Larin,  $\sim 50,000$  Feynman diagrams!



These are all in the class of zero or one kinematic variable.

# Major Advance of Past Few Years

Computations involving more than 1 kinematic variable is a new art.  
This is what we focus on here.



## Key to Progress

In the past few years the field of high loop computations has gotten a tremendous boost due to the influx of energetic bright young people.

Babis Anastasiou, Andrzej Czarnecki, Daniel de Florian, Thomas Gehrmann, Massimiliano Grazzini, Robert Harlander, Sven Heinemeyer, Bill Kilgore, Kirill Melnikov, Sven Moch, Zoltan Nagy, Carlo Oleari, Matthias Steinhauser, M.E. Tejeda-Yeomans, Peter Uwer, Doreen Wackeroth, Stefan Weinzierl, and many others

## The difficulties at NNLO

Every step in the construction of a physical cross-section involving two-loop amplitudes has major difficulties.

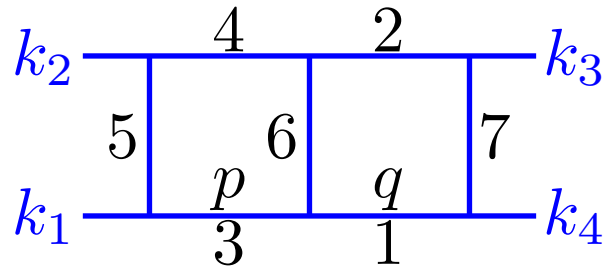
1. Loop integrals.
2. Scattering Amplitudes.
3. Infrared divergences and phase space integrals.
4. NNLO parton evolution (needed for Tevatron or LHC) via DGLAP equation.
5. Numerical programs for making quantitative predictions for experiments.

Remarkable progress in the past few years.

## Procedure for Loop Integration

- Cancel as many propagators as possible rewriting numerator factors as inverse propagators. Irreducible integrals remain.
- Schwinger (or Feynman) parameterize.
- Replace numerator parameters with higher powers of propagators.
- Construct system of ibp and Lorentz invariance identities.
- Solve system of equations in terms of *master integrals*.
- Evaluate master integrals, e.g. by constructing differential eqns.

**Example:** Consider the planar double box integral



$$I[P] \equiv \int d^D p \, d^D q \, \frac{P(p, q, k_i)}{p_1^2 p_2^2 \cdots p_7^2}$$

Using Schwinger parametrization  $\frac{1}{(p_i^2)^{\nu_i}} \sim \int_0^\infty dt_i t_i^{\nu_i-1} \exp(-t_i p_i^2),$

Integrate out loop momentum

$$I[1] \sim \int_0^\infty dt_1 dt_2 \dots dt_7 \prod_i t_i^{\nu_i-1} \Delta^{-D/2} \exp\left(i \frac{A}{\Delta} s + i \frac{t_5 t_6 t_7}{\Delta} t\right)$$

$$\Delta = (t_1 + t_2 + t_3)(t_3 + t_4 + t_5) + t_6(t_1 + t_2 + t_3 + t_4 + t_5 + t_7)$$

$$A = t_1 t_2 (t_3 + t_4 + t_5) + t_3 t_4 (t_1 + t_2 + t_7) + t_6 (t_1 + t_3) (t_2 + t_4)$$

Tensor integrals can be expressed in terms of scalar integrals but in different space-time dimensions and with shifted indices. — Tarasov (1996)

To integrate out loop momentum complete square:

$$p_1 \rightarrow Q + \frac{1}{\Delta} (-(t_4 t_6 + t_2(t_5 + t_6 + t_3 + t_4))(k_1 + k_2) - t_5 t_6 k_1 + t_7(t_5 + t_6 + t_3 + t_4)k_4)$$

After loop integration, left with  $1/\Delta$  which effectively changes  $D \rightarrow D + 2$  and each  $t_i$  counts as a higher power of propagator.

Thus, can reduce everything to scalar integrals with multiple propagators,

$$\int \frac{d^D p \, d^D q}{(p_1^2)^{\nu_1} (p_2^2)^{\nu_2} \cdots (p_7^2)^{\nu_7}}$$

with  $\nu_i = 0, 1, 2, \dots$  and in various dimensions.

Integration by parts identities:

Tkachov; Chetyrkin, Tkachov (1981)

$$0 = \int d^D p \, d^D q \, \frac{\partial}{\partial \ell^\mu} \frac{b^\mu}{(p_1^2)^{\nu_1} (p_2^2)^{\nu_2} \cdots (p_7^2)^{\nu_7}}$$
$$\ell^\mu = p^\mu, q^\mu, \quad b^\mu = p^\mu, q^\mu, k_i^\mu$$

Typical equation:

Can also have numerator factors.

$$s\nu_1 \mathbf{1}^+ = \nu_7 \mathbf{7}^+ \mathbf{2}^- + \nu_6 \mathbf{6}^+ (\mathbf{2}^- - \mathbf{4}^-) + \nu_1 \mathbf{1}^+ \mathbf{2}^- - (D - 2\nu_2 - \nu_1 - \nu_7 - \nu_6)$$

Also get equations from Lorentz invariance.

Gehrmann, Remiddi, hep-ph/9912329

More equations than unknowns, once you go out “far enough”.

Laporta and Remiddi, hep-ph/9602416

Solve equations in terms of ‘simplest’ integrals: get irreducible **master integrals**

Laporta, hep-ph/0102033

# Master Integrals

To obtain the master integrals there are various methods:

## 1. Mellin-Barnes representation

Smirnov, hep-ph/9905323, hep-ph/0007032, hep-ph/0011056, hep-ph/0111160

$$\frac{1}{(X+Y)^\nu} = \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{i\infty}^{i\infty} dw \frac{Y^w}{X^{\nu+w}} \Gamma(\nu+w) \Gamma(-w)$$

## 2. Nested sums

Moch, Uwer, Weinzierl, hep-ph/0110083, hep-ph/0207167

$$Z(n; m_1, \dots, m_k; x_1, \dots, x_k) = \sum_{n \geq i_1 > i_2 > \dots > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}$$

## 3. Construct differential equations for them.

Gehrmann, Remiddi hep-ph/9912329

Very important check: Comparison to numerical program.

Binoth and Heinrich, hep-ph/0004013

## Example of differential equation approach: one-loop box

$$I_4^{(1)} = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 (p - k_1)^2 (p - k_1 - k_2)^2 (p - k_1 - k_2 - k_3)^2}$$

$$k_3^\mu \frac{\partial}{\partial k_3^\mu} I_4^{(1)} = \int \frac{d^D k}{(2\pi)^D} \frac{2k_3 \cdot (p - k_1 - k_2 - k_3)}{p^2 (p - k_1)^2 (p - k_1 - k_2)^2 (p - k_1 - k_2 - k_3)^4}$$

Use reduction procedure to re-express in terms of master integrals.

Get equations such as:

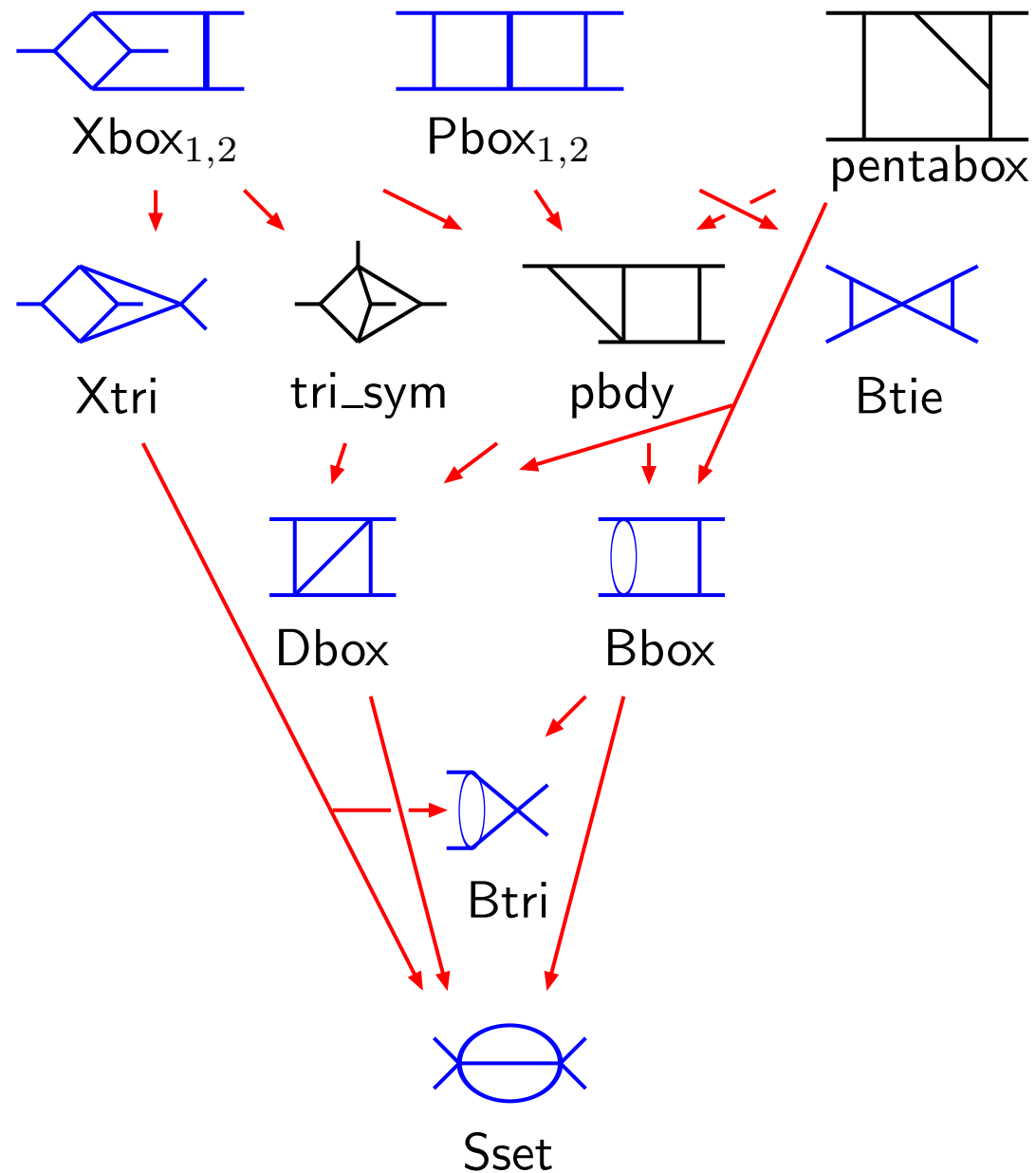
$$s \frac{\partial}{\partial s} I_4^{(1)} = -\frac{d-4}{2} I_4^{(1)} + \text{bubble integrals}, \quad s = (p_1 + p_2)^2$$

Same idea works at higher loops.

Differential equations straightforward to solve as a Laurent series in  $\epsilon$ , once the proper class of functions is identified.

Gehrmann, Remiddi hep-ph/9912329

# Two-loop integral inheritance chart



## Some References for Integrals

Pbox<sub>1,2</sub>, Bbox, Dbox: [Smirnov, hep-ph/9905323](#); [Smirnov & Veretin, hep-ph/9907385](#)

Xbox<sub>1</sub>, tri\_sym: [Tausk, hep-ph/9909506](#)

Xbox<sub>1,2</sub>, Xtri: [Anastasiou et al., hep-ph/0003261](#)

pentabox, Dbox: [Anastasiou, Glover, Oleari, hep-ph/9912251](#)

Bbox: [Anastasiou, Glover, Oleari, hep-ph/9907523](#)

### Some basic techniques:

- Integration by parts [Tkachov, PLB \(1981\)](#); [Chetyrkin & Tkachov, NPB \(1981\)](#)
- Lorentz invariance and differential equations [Gehrmann & Remiddi hep-ph/9912329](#)
- Systematic identification of master integrals [Laporta hep-ph/0102033](#)
- Nested sums [Moch, Uwer, Weinzierl hep-ph/0110083](#)

## Form of Results

In the case of two loops and massless  $2 \rightarrow 2$  scattering everything can be expressed in terms of standard polylogs up to 4th order.

$$\text{Li}_1(x) = -\ln(1-x), \quad \text{Li}_n(x) = \int_0^x dt \frac{\text{Li}_{n-1}(t)}{t}$$

Actually, Nielsen functions appear

$$S_{n,p}(x) = \frac{(-1)^{n-1+p}}{(n-1)!p!} \int_0^1 dt \frac{\ln^{n-1}(t) \ln^p(1-tx)}{t}$$

but for this case these can be re-expressed in terms of standard polylogs *e.g.*

$$\begin{aligned} S_{1,2}(-t/u) = & \frac{\pi^2}{6} \ln\left(\frac{u}{t+u}\right) - \frac{1}{6} \ln^3\left(\frac{u}{t+u}\right) - \frac{1}{2} \ln\left(\frac{t}{t+u}\right) \ln^2\left(\frac{u}{t+u}\right) \\ & - \ln\left(\frac{u}{t+u}\right) \text{Li}_2\left(\frac{t}{t+u}\right) - \text{Li}_3\left(\frac{u}{t+u}\right) + \zeta_3 \end{aligned}$$

# Harmonic Polylogarithms

As the number of mass scales increases this is insufficient.

For the case of a single external mass (e.g. as for  $e^+e^- \rightarrow 3$  partons) answer expressed in terms of generalizations called two-dimensional harmonic polylogs (Gehrmann and Remiddi).

The harmonic polylogs are:

$$H(1; x) = -\ln(1 - x), \quad H(0; x) = \ln x, \quad H(-1; x) = \ln(1 + x),$$

$$H(0, \dots, 0; x) = \frac{1}{w!} \ln^w x, \quad H(a, \vec{b}; x) = \int_0^x dt f(a; t) H(\vec{b}; t)$$

$$\text{where} \quad f(1; x) = \frac{1}{1 - x}, \quad f(0; x) = \frac{1}{x}, \quad f(-1; x) = \frac{1}{1 + x},$$

Set of fractions extended for 2 dimensional harmonic polylogs:

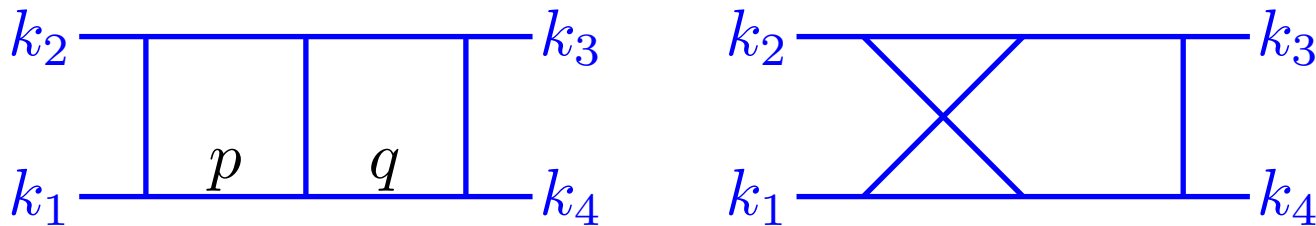
$$f(1 - z; x) = \frac{1}{1 - x - z}, \quad f(z; x) = \frac{1}{x + z}$$

# Construction of Amplitudes

Methods used to construct amplitudes:

- $D$ -dimensional unitarity: Tree amplitudes  $\rightarrow$  loop amplitudes.
- QGRAF to generate Feynman diagrams.
- Helicity method or interference method.

In either case loop integrals must be sorted in topological classes and relabeled.



A diagram can be represented by a collection of propagator momenta:

$$\{p, p - k_1, p - k_1 - k_2, p + q, q, q - k_3, q - k_3 - k_4\}$$

QGRAF labels related to integration labels via change of variables.

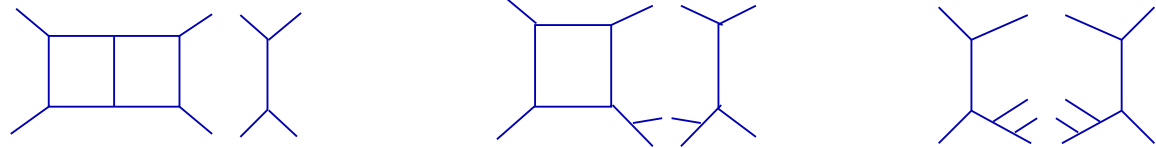
# New Two-loop Amplitudes

- Identical helicity QCD amplitude Bern, Dixon and Kosower (1999)
- Two-loop Bhabha scattering in QED,  $e^+e^- \rightarrow e^+e^-$ . Bern, Dixon and Ghinculov (2000)
- All two-loop  $2 \rightarrow 2$  QCD processes. Anastasiou, Glover, Oleari and Tejeda-Yeomans (2001)  
Bern, De Freitas, Dixon (2002)
- $\gamma\gamma \rightarrow \gamma\gamma$  Bern, Dixon, De Freitas, A. Ghinculov and H.L. Wong (2001)
- $gg \rightarrow \gamma\gamma$ . (Background to Higgs decay.) Bern, De Freitas, Dixon (2001)
- $\bar{q}q \rightarrow \gamma\gamma, \bar{q}q \rightarrow g\gamma, e^+e^- \rightarrow \gamma\gamma$  Anastasiou, Glover and Tejeda-Yeomans (2002)
- $e^+e^- \rightarrow 3$  partons Garland, Gehrmann, Glover, Koukoutsakis and Remiddi (2002)  
Moch, Uwer, Weinzierl (2002)
- DIS 2 jet and  $pp \rightarrow W, Z + 1$  jet Gehrmann and Remiddi (2002)

# Infrared Divergences

The source of endless grief in calculations are IR divergences.

All same order in  $\alpha_s$ :

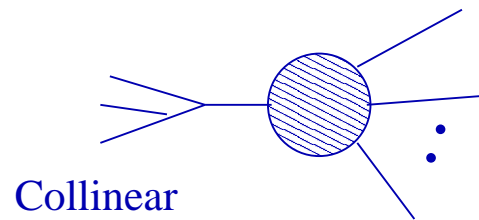


IR divergences cancel amongst contributions, but arise in intermediate steps.

1) Tree-level triple collinear and soft phase-space categorized.

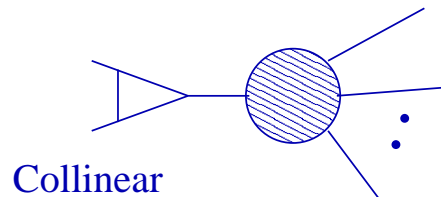
Campbell and Glover; Catani and Grazzini

Kosower; Weinzierl



2) At one-loop there are also non-trivial contributions.

Bern, Kilgore, Del Duca, and Schmidt; Kosower and Uwer; Catani and Grazzini



# Universal Two-loop Infrared Singularities

In a beautiful paper, [Stefano Catani \(1998\)](#) specified essentially the complete IR divergences of any two-loop QCD (and QED) process.

All IR divergences must cancel from a physical result:

real emission singularities + wizardry  $\longrightarrow$  two-loop IR divergences.

Catani's Magic Formula for IR divergences is extremely useful because:

- Substantial fraction of the answer for a two-loop amplitude is known prior to starting calculation.
- Provides a very stringent check on any calculation.
- Provides a way for organizing amplitudes.

Proof of structure of formula given by [Sterman and Tejeda-Yoemans](#).

# Catani's Magic Formula

Catani, hep-ph/9802439

$$\mathcal{M}_4(\alpha_s(\mu)) = 4\pi\alpha_s(\mu^2) \left[ \mathcal{M}_4^{(0)} + \frac{\alpha_s(\mu)}{2\pi} \mathcal{M}_4^{(1)} + \left( \frac{\alpha_s(\mu)}{2\pi} \right)^2 \mathcal{M}_4^{(2)} + \mathcal{O}(\alpha_s^3(\mu)) \right]$$

$$|\mathcal{M}_n^{(2)}\rangle_{\text{R.S.}} = \mathbf{I}^{(1)}(\epsilon) |\mathcal{M}_n^{(1)}\rangle_{\text{R.S.}} + \mathbf{I}_{\text{R.S.}}^{(2)}(\epsilon) |\mathcal{M}_n^{(0)}\rangle_{\text{R.S.}} + |\mathcal{M}_n^{(2)\text{fin}}\rangle_{\text{R.S.}}$$

$$\mathbf{I}^{(1)}(\epsilon) = \frac{1}{2} \frac{e^{-\epsilon\psi(1)}}{\Gamma(1-\epsilon)} \sum_{i=1}^n \sum_{j \neq i}^n \mathbf{T}_i \cdot \mathbf{T}_j \left[ \frac{1}{\epsilon^2} + \frac{\gamma_i}{\mathbf{T}_i^2} \frac{1}{\epsilon} \right] \left( \frac{\mu^2 e^{-i\lambda_{ij}\pi}}{2p_i \cdot p_j} \right)^\epsilon$$

$$\mathbf{I}_{\text{R.S.}}^{(2)}(\epsilon) = -\frac{1}{2} \mathbf{I}^{(1)}(\epsilon) \left( \mathbf{I}^{(1)}(\epsilon) + \frac{4\pi\beta_0}{\epsilon} \right) + \frac{e^{\epsilon\psi(1)} \Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left( \frac{2\pi\beta_0}{\epsilon} + K_{\text{R.S.}} \right) \mathbf{I}^{(1)}(2\epsilon) + \mathbf{H}_{\text{R.S.}}^{(2)}(\epsilon)$$

$\mathbf{I}^{(1)}(\epsilon)$  describes one-loop divergences.

$\mathbf{H}_{\text{R.S.}}^{(2)}(\epsilon)$  is renormalization-scheme dependent and has at most  $\frac{1}{\epsilon}$  poles.

# Parton Distribution Functions

For initial state protons one also needs parton distributions evolved using NNLO QCD.

There has been a large amount of work on this. So far only approximate solution.

van Neerven and Zijlstra (1993); Catani and Hautman (1994)

Larin, Nogueira, Retey, van Ritbergen, Vermaseren (1997)

van Neerven and Vogt (2000); Retey and Vermaseren (2001)

Moch, Vermaseren, and Vogt (2001)

Starting to be implemented in global fits (MRST) for pdfs.

Martin, Roberts, Stirling, Thorne (2002)

This is very important for true NNLO calculations.

Uncertainties in pdf's are now being quantified.

Giele, Keller and Kosower

Kuhlmann, *et al.*; J. Pumplin *et al.*

# Physical Predictions

Everything still needs to be put together in a numerical program for producing physical cross-sections.

In general, this is non-trivial, because of the IR divergences. At NLO general solutions to this problem exist. Giele, Glover and Kosower (1993)  
Frixione, Kunszt and Signer (1995); Catani and Seymour (1996)

Done at NNLO in special cases:

- Drell-Yan,  $W$  or  $Z$  production Hamberg, van Neerven and Matsuura (1991)
- Inclusive Higgs production at hadron colliders.  
Catani, de Florian and Grazzini (2001); Harlander and Kilgore (2002);  
Anastasiou and Melnikov (2002)

Recent progress on general IR subtraction terms. Kosower hep-ph/0212097;  
Wienzierl hep-ph/0302180

This problem still needs a complete solution.

## Recent Concrete Applications

Three very recent examples of concrete applications of the above two-loop progress:

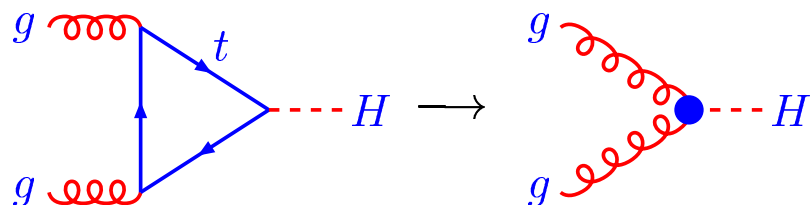
- Inclusive Higgs production at NNLO
- Improved understanding of the QCD background to Higgs production at the LHC (for a light Higgs).
- Theoretical study of  $N = 4$  super-Yang-Mills theory.

**This is just the beginning!**

These examples, bypass the technical problem with dealing with IR divergent phase space at NNLO.

# Inclusive Higgs Production

For a light mass Higgs use effective vertex:



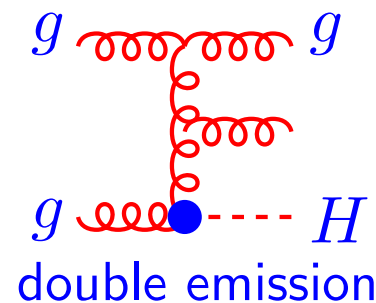
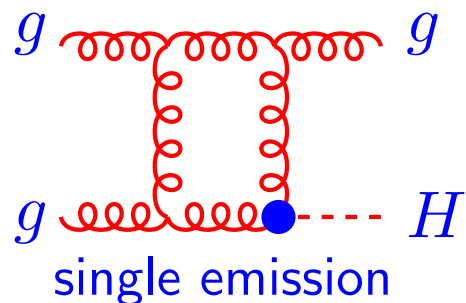
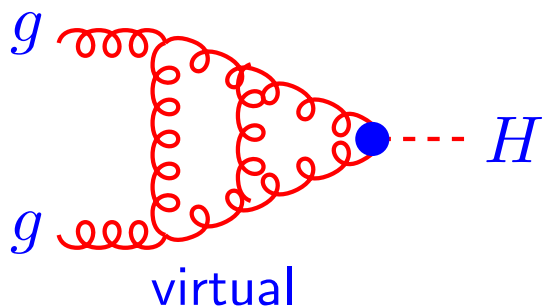
Wilzcek, 1977

Vainshtein et al., 1979

Loop corrections for the effective vertex computed through  $\mathcal{O}(\alpha_s^4)$

NNLO sampling of diagrams:

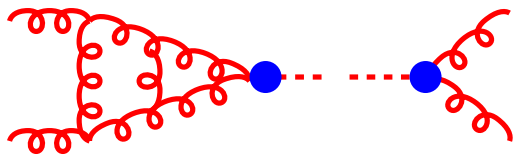
Chetyrkin et al., 1997



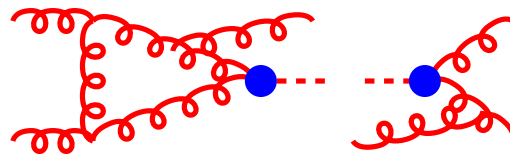
The difficult part of the calculation is dealing with the IR singular parts of the double real emission contributions. This was worked out by Harlander and Kilgore (PRL 2002) by expanding in  $(1 - M_H^2/\hat{s})$ .

# Application of New Technology

A useful application of the new integration technology has been to provide a straightforward way to obtain the **exact** phase space integration over the double real emission.



148 terms

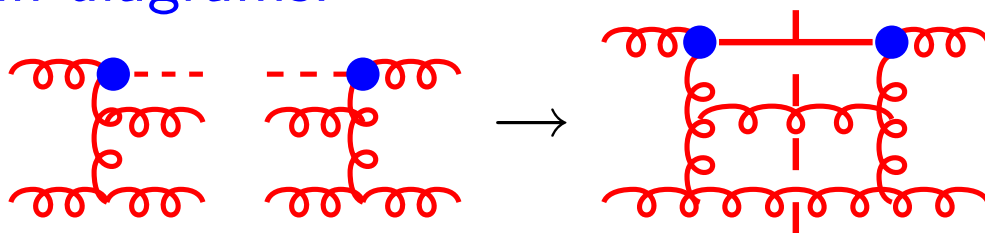


635 terms



594 terms

Use the optical theorem (unitarity) to promote the phase-space integral to the imaginary part of a forward scattering amplitude computable via Feynman diagrams:



Anastasiou and Melnikov  
(hep-ph/0207004)

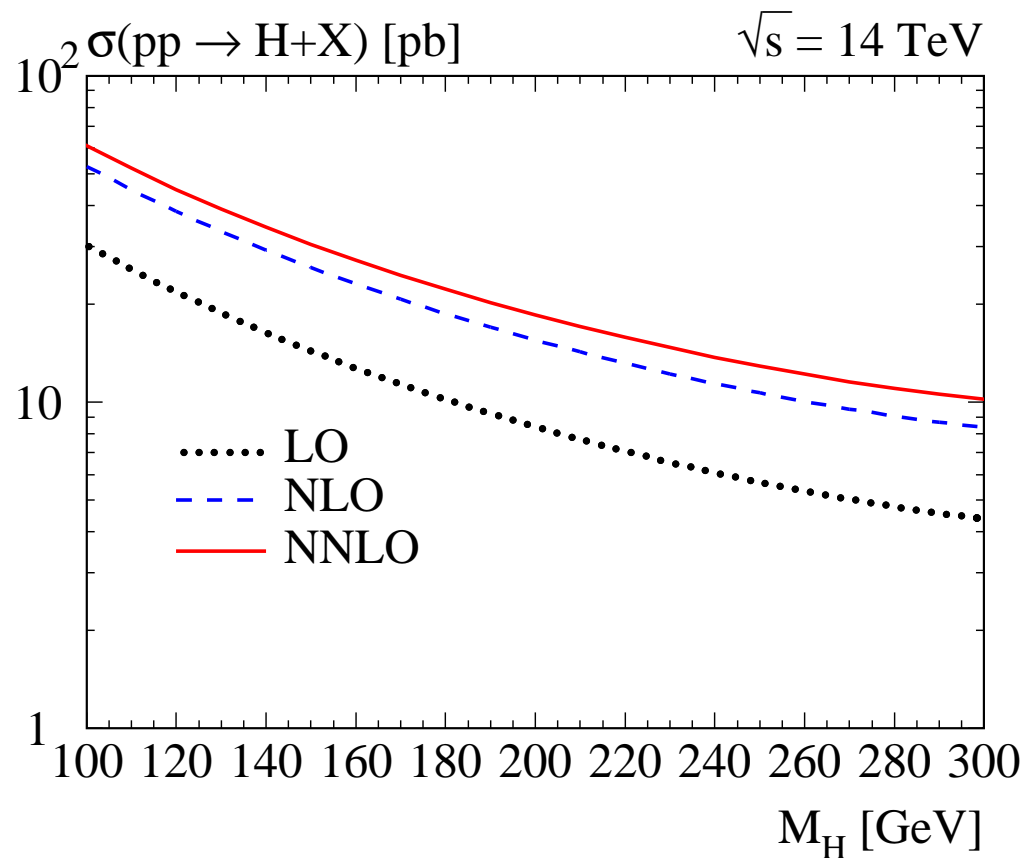
Use automated computer program (QGRAF) to produce Feynman diagrams. Then apply the previous integration technology.

Applicable to certain differential cross-sections.

Anastasiou, Dixon and Melnikov

# NNLO Inclusive Higgs Production at LHC

Harlander and Kilgore (hep-ph/0201206)  
Anastasiou and Melnikov (hep-ph/0207004)



Fact that the NNLO value is close to the NLO value suggests perturbation theory is under control. Result is also close to earlier approximate calculations of Catani, de Florian and Grazzini. Further refinements are coming.

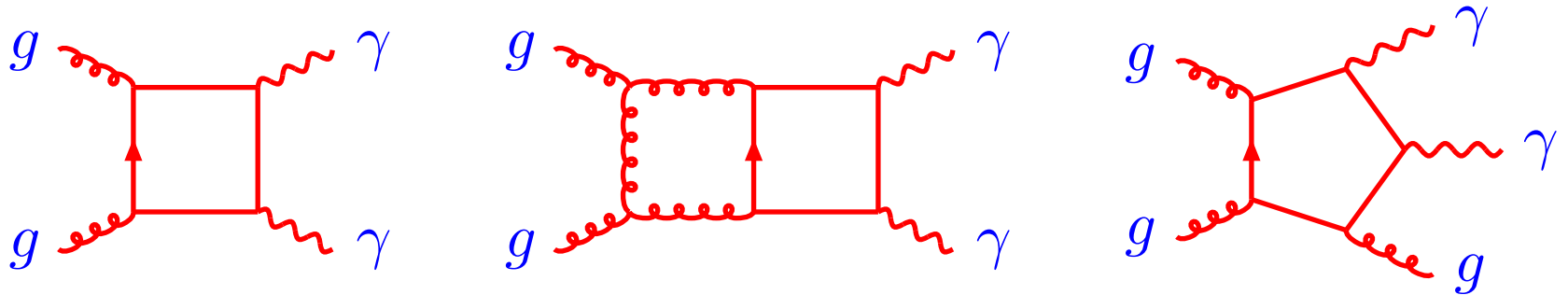
# LHC Higgs Search: background to $H \rightarrow \gamma\gamma$

For a low mass Higgs ( $M_H < 140$  GeV) the preferred search mode is via the rare decay  $H \rightarrow \gamma\gamma$ .

Leading and next-to-leading order QCD subprocesses for  $pp \rightarrow \gamma\gamma X$  form an irreducible background:



LHC is a glue factory, hence gluon fusion is important background:



NLO corrections to gluon fusion are sizable. Two-loop amplitude obtained via the new technology.

# Sample Two-Loop Finite Parts for

$$gg \rightarrow \gamma\gamma$$

Bern, De Freitas, Dixon

where

$$\begin{aligned}
 M_{--++}^{(2)} &= -\frac{3}{2}, \\
 M_{-+++}^{(2)} &= \frac{1}{8} \left[ \left( \frac{x^4}{y^2} + \frac{1}{y^2} - 2\frac{x^2}{y} + y^2 \right) X^2 - \frac{1}{2} (x^2 + y^2) (2XY - \pi^2) - 4 \left( \frac{x^2}{y} - 1 \right) X \right. \\
 &\quad \left. + 2i\pi \left[ \left( 1 - 2\frac{x}{y^2} \right) X + \frac{x}{y} - \frac{1}{x} \right] + \{t \leftrightarrow u\} \right], \\
 M_{++++}^{(2)} &= -2(x^2 + y^2) \text{Li}_4(-x) - (x - y) \text{Li}_4(-x/y) + 2x^2 X \left( \text{Li}_3(-x) + \text{Li}_3(-y) \right) \\
 &\quad + \frac{\pi^2}{6} (x - y) \text{Li}_2(-x) + x^2 \left( -\frac{1}{6} X^4 + \frac{2}{3} X^3 Y - \frac{\pi^2}{6} XY + \frac{4}{45} \pi^4 \right) \\
 &\quad - (3x + y) \left( \text{Li}_3(-x) - X \text{Li}_2(-x) \right) - \frac{1}{12} (9x - y) (X^3 - 3X^2 Y) \\
 &\quad + (3x + 5y) \frac{\pi^2}{12} X - 2\zeta_3 - \frac{1}{4} \left( \frac{x^4}{y^2} - x^2 + 4xy + 2y^2 \right) X^2 + \frac{1}{2} (3x + 4y) xXY \\
 &\quad - (7x + 10y) x \frac{\pi^2}{12} + \frac{1}{2} \left( \frac{x^2}{y} + 4x + y \right) X - \frac{1}{4} \\
 &\quad + i\pi \left[ 2(x^2 + y^2) \text{Li}_3(-x) - x^2 \left( \frac{1}{3} X(X^2 + \pi^2) - X^2 Y \right) \right. \\
 &\quad \left. + \frac{1}{2} (x - y) \left( \text{Li}_2(-x) - \text{Li}_2(-y) \right) - \frac{1}{2} (3x + y) X^2 + 2x \left( XY - \frac{\pi^2}{6} \right) \right. \\
 &\quad \left. + \frac{1}{2} \left( -\frac{x^2}{y^2} + 2\frac{x}{y} + 1 \right) X - \frac{x}{2y} - 1 \right] + \{t \leftrightarrow u\}, \\
 M_{+-+}^{(2)} &= 2 \left( 2\frac{x}{y^2} - 1 \right) \left( \text{Li}_4(-x/y) - \text{Li}_4(-y) - Y \left( \text{Li}_3(-x) - \zeta_3 \right) \right. \\
 &\quad \left. + \frac{1}{48} X^4 - \frac{1}{6} XY^3 + \frac{1}{24} Y^4 + \frac{\pi^2}{12} Y^2 - \frac{7}{360} \pi^4 \right) \\
 &\quad - 2 \left( 2\frac{x}{y} + 1 \right) \left( \text{Li}_4(-x) + \frac{\pi^2}{6} \text{Li}_2(-x) - \frac{\pi^2}{12} X^2 \right) - 2\frac{x^2}{y^2} \left( X \left( \text{Li}_3(-x) - \zeta_3 \right) + \frac{1}{24} X^4 + \frac{7}{90} \pi^4 \right) \\
 &\quad + \left( 2\frac{x}{y} - 1 \right) \left( \text{Li}_3(-x) - X \text{Li}_2(-x) - \frac{\pi^2}{3} Y \right) + 2 \left( 2\frac{x}{y} + 1 \right) \left( \text{Li}_3(-y) + Y \text{Li}_2(-x) + \frac{1}{2} XY^2 \right) \\
 &\quad - \frac{1}{12} \left( 10\frac{x}{y} + 1 \right) X^3 + \frac{\pi^2}{6} \left( 2\frac{x}{y} + 5 \right) X - \left( 2\frac{x}{y} + 3 \right) \zeta_3 + \frac{1}{4} \left( 2\frac{x^4}{y^2} + 6\frac{x^3}{y} + x^2 - 4xy - 2y^2 \right) X^2 \\
 &\quad + \frac{1}{2} (2x^2 - y^2) XY - \frac{1}{4} \left( 4x^2 + 8xy + 10y^2 + 6\frac{y^3}{x} + \frac{y^4}{x^2} \right) Y^2 - (4x^2 - 4xy - 5y^2) \frac{\pi^2}{12} \\
 &\quad + \frac{1}{2} (2x + y^2) \left( \frac{X}{y} + \frac{Y}{x} \right) - \frac{1}{2} \\
 &\quad + i\pi \left[ \frac{2}{y^2} \left( \text{Li}_3(-x) - \zeta_3 \right) + \frac{1}{6} \left( 2\frac{x}{y^2} - 1 \right) X^3 + \left( 2\frac{x}{y} + 3 \right) \text{Li}_2(-x) - \frac{3}{4} \left( 2\frac{x}{y} + 1 \right) X^2 \right. \\
 &\quad \left. - \frac{1}{2} \left( 2\frac{x}{y^2} + 3 \right) X - \frac{1}{2} \left( 2 + 4\frac{y}{x} + \frac{y^2}{x^2} \right) Y - \frac{1}{y} - \frac{y}{2x} \right].
 \end{aligned} \tag{2.7}$$

Here

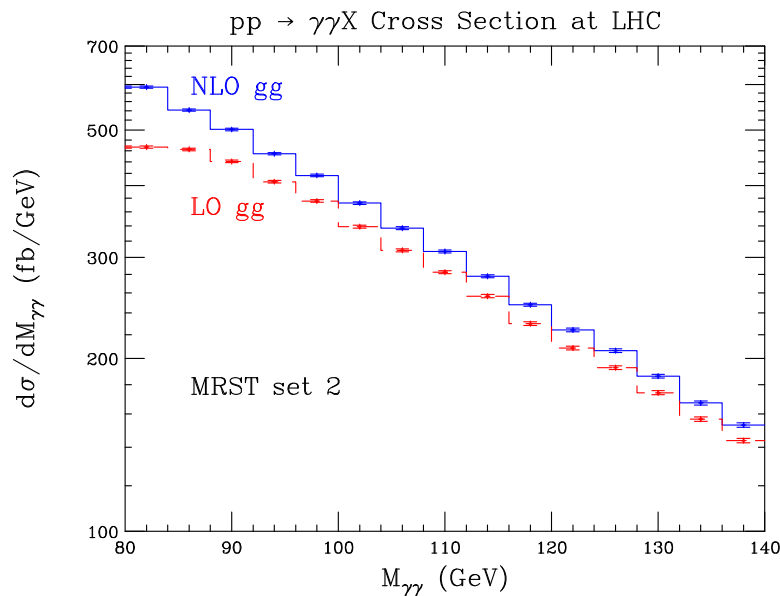
$$x \equiv \frac{t}{s}, \quad y \equiv \frac{u}{s}, \quad X \equiv \ln\left(-\frac{t}{s}\right), \quad Y \equiv \ln\left(-\frac{u}{s}\right). \tag{2.8}$$

6

Subleading color piece

# Phenomenological Implications

To obtain a physical cross-section the two-loop  $gg \rightarrow \gamma\gamma$  amplitude must be combined with the one-loop real emission diagrams as well as contributions with initial state quarks.



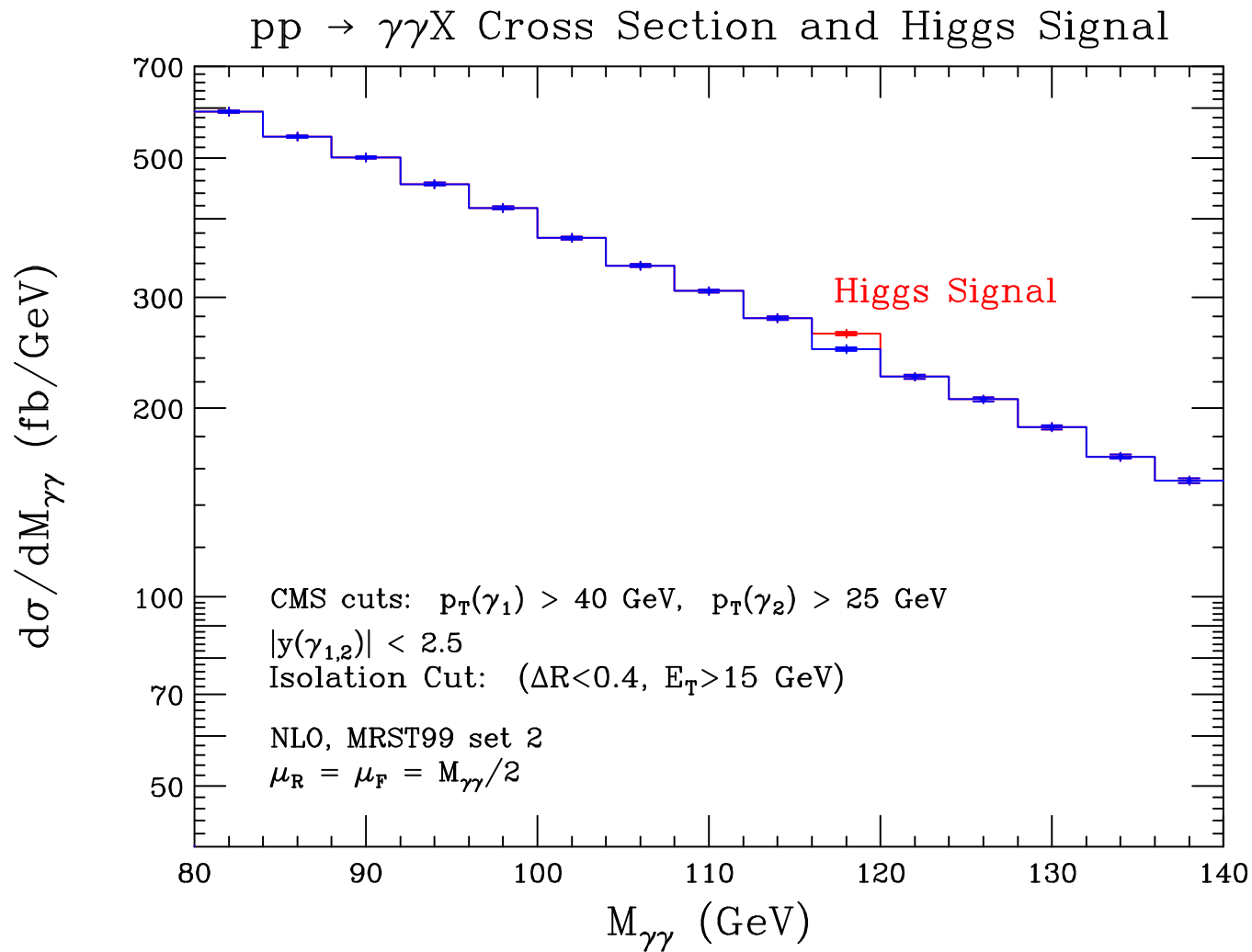
Berger, Braaten and Field (1984)  
Binoth, Guillet, Pilon, Werlen (2000)  
Bern, Dixon and Schmidt (2002)

$K$ -factor smaller than previous estimates.

$\sim 2$  years required to pull the Higgs signal out of the above background if  $m_H < 140$  GeV.

Can we improve the situation using quantitative theoretical predictions to find appropriate kinematic variables and cuts?

# The Higgs signal at the LHC

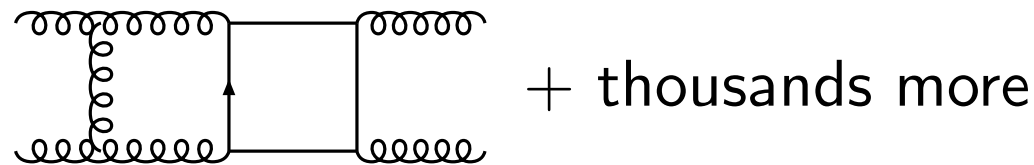


It will take about 2 years to pull signal out of background.

## Sample Theoretical Application

Wouldn't it be wonderful if we could find a Green function in a  $D = 4$  field theory that we could calculate to all loop orders?

Consider on-shell Green functions in  $N = 4$  super-Yang-Mills theory, e.g. at two loops:



We present evidence that this iterates to all loop orders.

As for QED, QCD, gravity and any other four dimensional theory with massless bosons these contain IR divergences.

We use dim. reg.

# On-shell Green functions to all loop orders

Some previous indications that the maximally supersymmetric Yang-Mills amplitudes can be evaluated to all orders:

- Maldacena conjecture suggests that  $D = 4, N = 4$  super-Yang-Mills should have *magical simplicity* in the planar limit.
- From the unitarity cuts, it has been known since 1997 that the four-point *integrand*s iterate to all loop orders. Bern, Yan, Rozowsky
- At tree level and at one loop, systematic formulae exist for arbitrary numbers of legs. This demonstrates that the analytic structure of the  $N = 4$  amplitudes is relatively simple. Bern, Dunbar, Dixon, Kosower

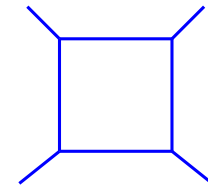
Here we will provide direct evidence at two loops that this intuition is correct.

# Two loops in terms of one loop

Z.B., Rozowsky, Yan

Anastasiou, Z.B., Dixon, Kosower

The four-point one-loop  $D = 4 - 2\epsilon, N = 4$  amplitude:



$$A_4^{1\text{-loop}}(s, t) = -st A_4^{\text{tree}} I_{1\text{-loop}}(s, t)$$

$$I^{1\text{-loop}}(s, t) \sim \frac{1}{st} \left[ \frac{2}{\epsilon^2} \left( (-s)^{-\epsilon} + (-t)^{-\epsilon} \right) - \ln^2 \left( \frac{t}{s} \right) - \pi^2 \right] + \mathcal{O}(\epsilon)$$

We also have the exact integral representation of the leading color two-loop amplitude:

$$A_4^{2\text{-loop}}(1, 2, 3, 4) = -st A_4^{\text{tree}}(1, 2, 3, 4) \left( s \mathcal{I}_4^{2\text{-loop}}(s, t) + t \mathcal{I}_4^{2\text{-loop}}(t, s) \right)$$

$$-st A_4^{\text{tree}} \left\{ s \begin{array}{c} 4 \text{---} \text{---} 1 \\ | \quad | \\ 3 \text{---} \text{---} 2 \end{array} + t \begin{array}{c} 4 \text{---} \text{---} 1 \\ | \quad | \\ 3 \text{---} \text{---} 2 \end{array} \right\}$$

The double box integral is a rather complicated object involving up to 4th order polylogarithms and Nielsen functions.

Nevertheless, for  $D \rightarrow 4 - \epsilon$  the two-loop planar amplitude undergoes an amazing simplification:

Anastasiou, Bern, Dixon, Kosower

$$M_4^{2\text{-loop}}(s, t) = \frac{1}{2} \left( M_4^{1\text{-loop}}(s, t) \right)^2 + f(\epsilon) M_4^{1\text{-loop}}(s, t) \Big|_{\epsilon \rightarrow 2\epsilon}$$

where

$$M_4^{n\text{-loop}} = A_4^{n\text{-loop}} / (N_c^n A_4^{\text{tree}}), \quad f(\epsilon) = \zeta_2 + \zeta_3 \epsilon - \frac{3}{2} \zeta_4 \epsilon^2$$

$f(\epsilon)$  is a universal IR function appearing in the Catani formula.

Thus, we have succeeded to express the two-loop amplitude as an iteration of the one loop amplitude. Can we show it continues?

Non-trivial polylogarithms and Nielsen function identities needed to demonstrate the above.

## Promise for Future

Other examples of what is on the horizon:

- $e^+e^- \rightarrow 3 \text{ jets}$  at NNLO. ‘Precision’ QCD at future linear collider  
*e.g.* 1% measurement of  $\alpha_s$ .  
Garland, Gehrmann, Glover, Koukoutsakis and Remiddi (2002)
- NNLO DIS + 2 jets.  
Gehrmann and Remiddi (2002)
- A complete NNLO 2 jet program for the Tevatron and LHC.  
Anastasiou, Glover, Oleari and Tejeda-Yeomans (2001)
- NNLO parton distribution functions.  
van Neerven *et al.*; Vermaseren *et al.*

For this promise to be realized, general algorithms for dealing with NNLO IR divergent phase space integration need to be set up.

The same technology is applicable to electroweak processes.

## Summary

1. Recent developments leading to new two-loop calculations. More will be forthcoming.
2. New technology already applied to produce physics results for inclusive Higgs production and to QCD background for Higgs production.
3. Theoretical study of  $N = 4$  super-Yang-Mills Green functions.
4. Potential for future: Unprecedented precision in high energy QCD and electroweak radiative corrections when more than a single kinematic invariant present.

We can be optimistic that this rapid pace of progress will continue!